

A Closed Form Characterization of the Stationary Outcome in Multilateral Bargaining

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Abstract

In this paper we consider infinite horizon multilateral bargaining with alternate offers. We prove that there exists only one stationary SPE outcome which is the limit outcome through finite truncation. We characterize this stationary SPE outcome in its closed form, and also extend the approach to multilateral bargaining with random order of moves

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1 Introduction

Rubinstein (1982) proved the existence of a unique subgame perfect equilibrium (SPE) in bilateral bargaining under complete information. This uniqueness property of SPE is lost when extending the model to bilateral bargaining under incomplete information or n -person bargaining under complete information¹ ($n \geq 3$). However, as pointed out in Sutton (1986), when restricting attention to stationary strategies or strategies that depends continuously on history, there will still be a unique SPE (Herrero (1985), and Binmore (1985)). That is, we have a unique stationary SPE outcome, which is also the unique SPE in the space of continuous strategies for multilateral bargaining with complete information. Furthermore, this unique stationary SPE outcome is the limit of the unique outcome in the finite period truncation game as $T \rightarrow \infty$.

However, it is not clear how to characterize this unique stationary outcome in a close form for agents with heterogeneous discount factors due to the inability to track agents' payoffs in the dynamic setting. A closed form characterization of this outcome might have meaningful benefits in the literature for two reasons. First, it would allow us to do comparative statics with ease. Second, it will provide a reduced form for applications of the multilateral bargaining model to real world problems.

The purpose of this paper is to characterize a closed form solution for the unique stationary SPE in the multilateral bargaining model with het-

¹For example, as shown in Sutton (1986), for 3-person bargaining problem with complete information and homogeneous discount factor δ , any division $x = (x_1, x_2, x_3)$ in the simplex can be supported by a SPE when the discount factor $\delta > \frac{1}{2}$.

erogeneous discount factors.

It should be noted that attempts has been made to generalize the bilateral model to multilateral settings with complete information by Dutta and Gevers (1984) as illustrated in Moulin (1986) and Sutton (1986). However, players in their setup are assumed to have identical discount factor due to the tractability problem.

The multiplicity of SPE is illustrated nicely in Sutton (1986), together with very insightful summary about the potential difficulties along the line of incomplete information or multi-person bargaining. And this is further demonstrated in Cai (2000). Since the procedure of subgame perfection no longer has the teeth to bite down the multiplicity of equilibria. It might be an interesting problem to characterize the unique stationary outcome in closed form.

In this paper, we study infinite horizon multilateral bargaining with alternate offers, denoted G^∞ , under complete information. We truncate G^∞ after finite period T with some disagreement payoff x which is imposed if no agreement is reached at the ending time T . Denoting this truncated game by $G^T(x)$.

We first characterize the SPE payoff of $G^T(x)$. Then we prove that the $\text{SPE}(G^T(x))$ converges to the unique stationary SPE outcome as $T \rightarrow \infty$, independent of the termination payoff of x . We further prove that the unique stationary SPE outcome corresponds to a unique fixed point of a column stochastic matrix which we call the bargaining operator B . This bargaining operator is constructed using agents' discount factors. Applying matrix theory allows us to calculate unique fixed point in its closed form. The

share of a player in G^∞ depends on one of the principle minors and the first order derivative of the characteristic polynomial of the bargaining operator, B .

Another problem in bargaining pointed out by Sutton (1986) is the existence of first mover advantage, where the player who moves first will obtain a larger share of the pie in the SPE outcome. It was suggested to randomize who moves first. In the extension we consider a more general set up by considering a randomization in each period by assuming the order of moves in each period follow a stationary Markov chain with some transition matrix P . We can extend our approach to this setting, and characterize the unique stationary SPE outcome.

The rest of the paper is organized as follows. Section 2 introduces the model of multilateral bargaining. We define the bargaining operator for each player, and link it to a homogeneous Markov chain using individual bargaining operators. It is shown to be irreducible, thus there is a unique invariant measure corresponding to the stationary outcome of the game G^∞ . In Section 3 we characterize the subgame perfect equilibrium for the multilateral bargaining model. We show that the unique subgame perfect equilibrium for the *finite* horizon case can be written using the bargaining operator that we introduced in Section 2. In the *infinite* horizon case the unique stationary subgame perfect equilibrium outcome is simply the fixed point of the bargaining operator. Section 4 illustrates the results with two examples, trilateral bargaining and multilateral bargaining with identical discount factors. Section 5 provides a closed-form characterization of the bargaining outcome for the general multilateral bargaining with heterogeneous discount factors.

Section 6 generalizes the results to multi-person bargaining with random order of moves where the right to make an offer follows a Markov chain. Section 7 concludes the paper.

2 Multilateral Model of Bargaining

The purpose of this section is to generalize Rubinstein's 2-person bargaining equilibrium to the n -person case with heterogeneous discount factors.

2.1 Description of the Game

Following Rubinstein 1982, we focus on the subgame perfect equilibrium with complete information.

There is a unit of surplus to be shared among a set of n individuals.

Time t runs from $0, 1, 2, \dots, T, \dots$ where $T \in \mathbb{N}$.

Starting from player 1, the players take turns offering a plan $x = (x_1, x_2, \dots, x_n)$ about how the pie is shared. Player i makes an offer at dates $t_i = \{t | t = (i - 1) \pmod{n}\}$.

Once the offer is accepted by all, the pie is shared accordingly. If the offer from player i is rejected by any of the other players, the game goes to the next date and it is player $(i + 1)$'s turn to make an offer. The individual discount factor is δ_i for player i .

To characterize the equilibrium outcome in G^T and G^∞ , we shall use the theory of stochastic matrices to help us write out the closed form solution of the stationary equilibrium outcome. This particular matrix, which we call the bargaining operator, is defined next.

2.2 The Bargaining Operator

We need at first to introduce the concept of stochastic matrix which plays central role in characterizing the unique equilibrium outcome in a closed form. Note that a column stochastic matrix is a matrix whose elements are nonnegative and the sum of elements in each column equals to 1.

Let $\Delta^n = \{x = (x_1, \dots, x_n) \mid x_i \geq 0, \sum x_i = 1\}$ be the n -simplex. Thus Δ^n is the set of feasible divisions of the pie in the n -person bargaining problem.

Each stochastic matrix leaves $\Delta^n = \{x \in R^n \mid x_i \geq 0, \sum x_i = 1\}$ invariant, thus has fixed points in Δ^n . When the matrix is positive, there will be a unique fixed point. We refer readers to the appendix for the related definitions and results.

Given player i 's discount factor δ_i , $i=1, \dots, n$, let

$$B_i = \begin{bmatrix} \delta_1 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \delta_2 & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1 - \delta_1 & 1 - \delta_2 & \dots & 1 - \delta_{i-1} & 1 & 1 - \delta_{i+1} & \dots & 1 - \delta_n \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \delta_n \end{bmatrix} = [b_{kl}^{(i)}]_{n \times n}$$

be the individual bargaining operator for player i , which is a matrix with the diagonal elements $b_{jj} = \delta_j$, for $j \neq i$, the i th row $b_{ij} = 1 - \delta_j$, for $j \neq i$, and $b_{ii} = 1$. And

$$B = B_1 B_2 \dots B_n$$

is the bargaining operator for the game, i.e., the composite of those individual B 's.

The next proposition describes properties of the above bargaining operator which is important to pin down the unique stationary SPE outcome.

Proposition 1 *i) Each of the individual bargaining operators B_i is a column stochastic operator which leaves Δ^n invariant.*

ii) $B = B_1 B_2 \dots B_n$ is a positive stochastic operator .

iii) The fixed point of B in Δ^n , x^ , is unique, and furthermore $x^* = \lim_{k \rightarrow \infty} B^k x^T$ for any initial condition of the backward induction $x^T \in \Delta^n$.*

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Proof. *i)* Simple observation shows $\sum_k b_{kl}^{(i)} = 1, \forall x \in \Delta^n, \sum_k (B_i x)_k = \sum_k \left(\sum_l b_{kl}^{(i)} x_l \right) = \sum_l \left(\sum_k b_{kl}^{(i)} \right) x_l = \sum_l x_l = 1$. Combining with nonnegativity of B_i , we have that B_i leaves Δ^n invariant, for $i = 1, 2, \dots, n$.

ii) It is easy to see that the (i, j) th element of B is

$$\begin{aligned} b_{ij} &= \sum_{h_{n-1}} \dots \sum_{h_2} \sum_{h_1} b_{ih_1}^{(1)} b_{h_1 h_2}^{(2)} \dots b_{h_{n-1} j}^{(n)} \\ &> b_{ij}^{(1)} b_{jj}^{(2)} \dots b_{jj}^{(n)} \\ &= (1 - \delta_j) \delta_j \dots 1 \dots \delta_j \\ &= \begin{cases} \delta_j^{n-2} (1 - \delta_j) > 0 & i \neq j \\ \delta_j^{n-1} > 0 & i = j \end{cases} \end{aligned}$$

Therefore B is a positive operator.

Clearly B is a column stochastic operator because each of the factors

B_i is column stochastic.

$$\begin{aligned}
\sum_i b_{ij} &= \sum_i \left[\sum_{h_{n-1}} \dots \sum_{h_2} \sum_{h_1} b_{ih_1}^{(1)} b_{h_1 h_2}^{(2)} \dots b_{h_{n-1} j}^{(n)} \right] \\
&= \sum_{h_{n-1}} \dots \sum_{h_2} \sum_{h_1} \left(\sum_i b_{ih_1}^{(1)} \right) b_{h_1 h_2}^{(2)} \dots b_{h_{n-1} j}^{(n)} \\
&= \sum_{h_{n-1}} \dots \sum_{h_2} \left(\sum_{h_1} b_{h_1 h_2}^{(2)} \right) \dots b_{h_{n-1} j}^{(n)} \\
&= \dots \\
&= \sum_{h_{n-1}} b_{h_{n-1} j}^{(n)} \\
&= 1.
\end{aligned}$$

iii) We have shown that B is a positive stochastic operator which is obviously a primitive. This implies that B is irreducible. Therefore, B has a unique fixed point in the simplex Δ^n , which corresponds to the unique eigenvector associated with the largest eigenvalue 1 (**please see appendix for relevant definitions**). ■

3 Characterization of Stationary Subgame Perfect Equilibrium Outcome

Given the results obtained in the last section, we are now ready to characterize the equilibrium outcome for the multilateral bargaining game.

3.1 The Finite Horizon Case

We first characterize the finite horizon bargaining equilibrium.

The finite horizon game $G^T(x^T)$ is the finite truncation of the ∞ – *horizon* game. In this finite truncation, players make alternate offers until

one offer is accepted by everybody. However, if no agreement is reached before or at T , then an exogenous division x^T is imposed on the players.

The following proposition shows that there is a subgame perfect outcome in the finite horizon bargaining game, and it can be written in a closed form using the bargaining operators.

Proposition 2 *i) If x is an acceptable offer made by player i in period t , then $x' = B_{i-1}x$ is an acceptable offer made by player $(i-1)$ in period $(t-1)$, (with the understanding that player 0 is player n).*

ii) $G^T(x^T)$ has a unique SPE outcome

$$x^* = B^{\lfloor \frac{T}{n} \rfloor} B_1 B_2 \dots B_{l-1} x^T,$$

where $T = l(\text{mod}(n))$, and $\lfloor \frac{T}{n} \rfloor$ is the integer part of $\frac{T}{n}$.

Proof. *i)* It is clear that if player i 's offer x is acceptable to all players, then $[B_{i-1}x]_j$ is indifferent from x_j for $j \neq i-1$, and for $j = i-1$, we have that $[B_{i-1}x]_{i-1}$ strictly preferable to x_{i-1} . Therefore $x' = B_{i-1}x$ is acceptable to all players.

ii) Since x^T is the only acceptable offer to all players in the last period, backward induction implies that the only acceptable offer in period $T-1$ is $B_{l-1}x^T$. By induction, it is easily shown that $x^* = B^{\lfloor \frac{T}{n} \rfloor} B_1 B_2 \dots B_{l-1} x^T$ is the only acceptable offer in the first period. Thus there is only one subgame perfect equilibrium with the given payoff. ■

3.2 The Infinite Horizon Case

In this section we aim to fully characterize the efficient subgame perfect equilibrium in the game G^∞ .

The following proposition shows that for the multilateral bargaining model there is only one agreement outcome in Δ^n that could be supported as the stationary subgame perfect equilibrium for G^∞ .

Proposition 3 *i) There is a subgame perfect equilibrium outcome $x^* \in \Delta^n$ for G^∞ , which is a fixed point for $B = B_1 \cdots B_n$ in Δ^n .*

iii) For any $x^0 \in \Delta^n$, the sequence $x^k = B^k x^0$ converges to the unique stationary bargaining outcome x^ . Hence it is the unique limiting point for SPE payoff in $G^T(x)$, for all x .*

Proof. Consider a subgame G_1^∞ , which is the subgame starting in period $t = n$, with player 1 making the offer again. It should be noted that this game is isomorphic to the original game. Therefore the set of agreements that could be supported as a stationary subgame perfect equilibrium should be identical for G^∞ and G_1^∞ . That is, $E_{G^\infty} = E_{G_1^\infty} \subset \Delta$. Let G_k^∞ be the subgame starting in period $t = kn$ with player 1 making the offer. Then G_k^∞ is isomorphic to G^∞ , and $E = E_{G^\infty} = E_{G_k^\infty} \subset \Delta$, for $k = 1, 2, \dots$.

On the other hand, subgame perfection tells us that $x' \in E_{G^\infty}$ is a subgame perfect equilibrium outcome if and only if there is a subgame perfect outcome $x \in E_{G^1}$ such that $x' = Bx$. This implies that $E = B(E) = \dots = B^k(E)$, for $k = 1, 2, \dots$.

Since B is an irreducible stochastic operator, B has a unique fixed point x^* in Δ^n , and furthermore $x^* = \lim_{k \rightarrow \infty} B^k x^0$ for any $x^0 \in \Delta^n$. Therefore $E = \{x^*\} = \lim_{k \rightarrow \infty} B^k(\Delta^n)$. ■

Note that the above result illustrates what agreement in Δ^n can be supported as the stationary subgame perfect equilibrium. However,

it should be noted that when $n \geq 3$, the set of SPE outcome can be very large (see footnote 1). The above proposition tells us the only SPE outcome in G^∞ that can be approached by SPE outcome of G^T is the fixed point of the bargaining operator B .

4 Two Examples

In this section we illustrate the results that we obtained with two examples.

4.1 Example 1: Trilateral Bargaining

Let $\delta_1, \delta_2, \delta_3$ be the discount factors of the three players, respectively. We have the following bargaining operator for each of them:

$$B_1 = \begin{bmatrix} 1 & 1 - \delta_2 & 1 - \delta_3 \\ 0 & \delta_2 & 0 \\ 0 & 0 & \delta_3 \end{bmatrix}, \quad B_2 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 1 - \delta_1 & 1 & 1 - \delta_3 \\ 0 & 0 & \delta_3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} \delta_1 & 0 & 0 \\ 0 & \delta_2 & 0 \\ 1 - \delta_1 & 1 - \delta_2 & 1 \end{bmatrix}.$$

And $B = B_1 B_2 B_3$ is the bargaining operator for the trilateral game. This operator has a unique fixed point in Δ^3 (the eigenvector associated with the largest eigenvalue 1 of B), which is the unique subgame perfect equilibrium outcome:

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} \frac{(1-\delta_2)(1-\delta_3)(1+\delta_3+\delta_2\delta_3)}{(1-\delta_1\delta_2\delta_3)^2 + \delta_1\delta_3(\delta_2-\delta_3) + \delta_1\delta_2(\delta_3-\delta_1) + \delta_2\delta_3(\delta_1-\delta_2)} \\ \frac{\delta_2(1-\delta_1)(1-\delta_3)(1+\delta_1+\delta_1\delta_3)}{(1-\delta_1\delta_2\delta_3)^2 + \delta_1\delta_3(\delta_2-\delta_3) + \delta_1\delta_2(\delta_3-\delta_1) + \delta_2\delta_3(\delta_1-\delta_2)} \\ \frac{\delta_3^2(1-\delta_1)(1-\delta_2)(1+\delta_2+\delta_1\delta_2)}{(1-\delta_1\delta_2\delta_3)^2 + \delta_1\delta_3(\delta_2-\delta_3) + \delta_1\delta_2(\delta_3-\delta_1) + \delta_2\delta_3(\delta_1-\delta_2)} \end{bmatrix}.$$

When the three players have the same discount factor $\delta_1 = \delta_2 = \delta_3 = \delta$, we have

$$B = \begin{bmatrix} 1 - \delta + \delta^3 & 1 - \delta & 1 - \delta \\ \delta(1 - \delta) & \delta(1 - \delta + \delta^2) & \delta(1 - \delta) \\ \delta^2(1 - \delta) & \delta^2(1 - \delta) & \delta^2 \end{bmatrix}.$$

And the equilibrium outcome in this case is simplified to

$$\begin{bmatrix} x_1^* \\ x_2^* \\ x_3^* \end{bmatrix} = \begin{bmatrix} \frac{1}{1+\delta+\delta^2} \\ \frac{\delta}{1+\delta+\delta^2} \\ \frac{\delta^2}{1+\delta+\delta^2} \end{bmatrix}.$$

4.2 Example 2: N -person Bargaining with Identical Discount Factor

Now we illustrate one more example where the n players have identical discount factor $\delta \in (0, 1)$. In this case the bargaining operator for player i is

$$B_i = \begin{bmatrix} \delta & 0 & \dots & 0 & 0 & 0 & \dots & 0 \\ 0 & \delta & \dots & 0 & 0 & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 1-\delta & 1-\delta & \dots & 1-\delta & 1 & 1-\delta & \dots & 1-\delta \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 0 & 0 & \dots & \delta \end{bmatrix}, \quad i = 1, 2, \dots, n.$$

And the bargaining operator for the game is

$$\begin{aligned} B &= B_1 B_2 \dots B_n \\ &= \begin{bmatrix} 1-\delta+\delta^n & 1-\delta & 1-\delta & \dots & 1-\delta & 1-\delta \\ \delta-\delta^2 & \delta-\delta^2+\delta^n & \delta-\delta^2 & \dots & \delta-\delta^2 & \delta-\delta^2 \\ \delta^2-\delta^3 & \delta^2-\delta^3 & \delta^2-\delta^3+\delta^n & \dots & \delta^2-\delta^3 & \delta^2-\delta^3 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \delta^{n-2}-\delta^{n-1} & \delta^{n-2}-\delta^{n-1} & \delta^{n-2}-\delta^{n-1} & \dots & \delta^{n-2}-\delta^{n-1}+\delta^n & \delta^{n-2}-\delta^{n-1} \\ \delta^{n-1}-\delta^n & \delta^{n-1}-\delta^n & \delta^{n-1}-\delta^n & \dots & \delta^{n-1}-\delta^n & \delta^{n-1} \end{bmatrix}. \end{aligned}$$

This is a positive (column) stochastic matrix that has a unique fixed point in Δ^n . This fixed point is the only subgame perfect equilibrium for the game G^∞ :

$$x^* = \left(\frac{(1-\delta)}{1-\delta^n}, \frac{\delta(1-\delta)}{1-\delta^n}, \frac{\delta^2(1-\delta)}{1-\delta^n}, \dots, \frac{\delta^{n-1}(1-\delta)}{1-\delta^n} \right).$$

5 A Closed-form Characterization for the Stationary SPE payoffs

In this section we provide a closed-form representation for the stationary n -person bargaining outcome.

The following lemma is well-known in matrix theory (see Lancaster and Tismenetsky, 1985, p.548):

Lemma 1 *If B is a stochastic matrix, then $\lim_{n \rightarrow \infty} B^n$ exists if and only if B is primitive. In this case, $\lim_{n \rightarrow \infty} B^n = \frac{A(1)}{c^{(1)}(1)}$, where $A(\lambda) = \text{adj}(\lambda I - B)$, the adjoint matrix for $(\lambda I - B)$, and $c(\lambda) = \det(\lambda I - B)$ is the characteristic polynomial for B , and $c^{(1)}(1)$ is the first-order derivative evaluated at $\lambda = 1$.*

Partition the bargaining operator for player k as $B_k = \left(b_{ij}^{(k)}\right)_{n \times n} = \begin{bmatrix} B_{11}^{(k)} & B_{12}^{(k)} \\ B_{21}^{(k)} & B_{22}^{(k)} \end{bmatrix}$, where $B_{11}^{(k)}$ is a scalar matrix and $B_{22}^{(k)}$ is $(n-1) \times (n-1)$. And the bargaining operator for the game, $B = B_1 B_2 \dots B_n$, is partitioned accordingly, $B = (b_{ij})_{n \times n} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$.

We define the following share function $S(\delta_2, \delta_3, \dots, \delta_n) = \det(I - B_{22})$.

We have the following lemma.

Lemma 2 $S(\delta_2, \delta_3, \dots, \delta_n) = \det(I - B_{22}) = \det(I - B_{22}^{(1)} B_{22}^{(2)} \dots B_{22}^{(n)})$ is a polynomial function of $(\delta_2, \delta_3, \dots, \delta_n)$, independent of player 1's discount factor δ_1 .

Proof. Given the partition $B_k = \left(b_{ij}^{(k)}\right)_{n \times n} = \begin{bmatrix} B_{11}^{(k)} & B_{12}^{(k)} \\ B_{21}^{(k)} & B_{22}^{(k)} \end{bmatrix}$, for $k = 2, \dots, n$, we have $B_{12}^{(k)} = \mathbf{0}$. This implies the $(2, 2)$ block for B , $B_{22} = B_{22}^{(1)} B_{22}^{(2)} \dots B_{22}^{(n)}$. None of the factors in this matrix product depends on δ_1 . Therefore $\det(I - B_{22})$ is a polynomial with variables $(\delta_2, \delta_3, \dots, \delta_n)$. ■

Let $S_k(\delta_1, \dots, \widehat{\delta}_k, \dots, \delta_n) = S(\delta_{k+1}, \dots, \delta_n, \delta_1, \dots, \delta_{k-1})$ be the polynomial function obtained by rotating the variables. It is a polynomial function of $(n-1)$ variables $(\delta_1, \dots, \delta_{k-1}, \delta_{k+1}, \dots, \delta_n)$.

Now we are ready to state the following closed-form solution for the n -person bargaining problem.

Proposition 4 *In the unique stationary bargaining outcome $x^* = (x_1^*, \dots, x_n^*)$, the share for player k can be represented as $x_k^* = \frac{\delta_k^{k-1} S_k(\delta_1, \dots, \widehat{\delta}_k, \dots, \delta_n)}{c^{(1)}(1)}$, where $S_k(\delta_1, \dots, \widehat{\delta}_k, \dots, \delta_n) = S(\delta_{k+1}, \dots, \delta_n, \delta_1, \dots, \delta_{k-1}) = \det(I - B_{22}^{(k)} \dots B_{22}^{(n)} B_{22}^{(1)} \dots B_{22}^{(k-1)})$ is defined as above.*

Proof. From Lemma 2 we know that $\lim_{n \rightarrow \infty} B^n = \frac{A(1)}{c^{(1)}(1)}$, and $B \frac{A(1)}{c^{(1)}(1)} = \lim_{n \rightarrow \infty} B^{n+1} = \frac{A(1)}{c^{(1)}(1)}$. Therefore each non-zero column of $\frac{A(1)}{c^{(1)}(1)}$ must be an eigenvector of B associated with the unitary eigenvalue. That is, each nonzero column of $\frac{A(1)}{c^{(1)}(1)}$ must be equal to the bargaining outcome since B is a primitive matrix and the unitary eigenvalue is a simple root.

Next we examine the $(1, 1)$ element of the matrix $\frac{A(1)}{c^{(1)}(1)}$, which is $\frac{\det(I - B_{22})}{c^{(1)}(1)}$, where B_{22} is the $(n-1) \times (n-1)$ block in the bargaining matrix B . Since $B_{22} = B_{22}^{(1)} B_{22}^{(2)} \dots B_{22}^{(n)}$, each of $B_{22}^{(2)}, \dots, B_{22}^{(n)}$ is a $(n-1) \times (n-1)$ stochastic matrix, and $B_{22}^{(1)} = \text{diag}(\delta_2, \delta_3, \dots, \delta_n)$ is a matrix with spectral radius less than 1. Then $B_{22} = B_{22}^{(1)} B_{22}^{(2)}, \dots, B_{22}^{(n)}$ is a matrix with spectral radius less than 1. Therefore $\det(I - B_{22}) \neq 0$. This means the first column of $\frac{A(1)}{c^{(1)}(1)}$ is non-zero, and it must be identical to the bargaining outcome x^* . Therefore the first player's share is the $(1, 1)$ element of $\frac{A(1)}{c^{(1)}(1)}$, which equals $\frac{\det(I - B_{22})}{c^{(1)}(1)} = \frac{S(\delta_2, \delta_3, \dots, \delta_n)}{c^{(1)}(1)}$.

Recalling that the game is homogeneous, in the second period when

player 2 is the leading player, the new bargaining matrix is given by $\tilde{B} = QB_2 \dots B_n B_1 Q^T$. Where $Q = \begin{bmatrix} 0_{(n-1) \times 1} & I_{(n-1) \times (n-1)} \\ 1 & 0_{1 \times (n-1)} \end{bmatrix}$, Since $B = (B_1 P^T) \tilde{B} (B_1 P^T)^{-1}$, i.e., \tilde{B} is similar to B , therefore \tilde{B} and B share the same characteristic polynomial, $c_{\tilde{B}}(\lambda) = c_B(\lambda)$. This implies their derivatives must be the same: $c'_{\tilde{B}}(1) = c'_B(1) = c^{(1)}(1)$.

Finally notice that $\det(I - \tilde{B}_{22}) = \det(I - B_{22}^{(2)} \dots B_{22}^{(n)} B_{22}^{(1)}) = S(\delta_3, \dots, \delta_n, \delta_1)$. We know that player 2's share is $\frac{S(\delta_3, \dots, \delta_n, \delta_1)}{c^{(1)}(1)}$, if he were to move first. This implies his share in the game where he moves second would be $x_2^* = \frac{\delta_2 S(\delta_3, \dots, \delta_n, \delta_1)}{c^{(1)}(1)}$, where δ_2 is his discount factor. The same reasoning implies that the share by player k would be written as $x_k^* = \frac{\delta_k^{k-1} S(\delta_{k+1}, \dots, \delta_n, \delta_1, \dots, \delta_{k-1})}{c^{(1)}(1)}$, completing the proof. ■

Up to now we have been assuming that the order of moves is exogenously given in a fixed fashion. In the next section we relax this assumption by allowing the order of moves to be random, and this random transition of the right to make offers is governed by a Markov chain with transition matrix P .

6 Multilateral Bargaining with Randomized Order of Moves

In this section, the order of moves is allowed to be random. Suppose player i is the player who makes an offer in period t . Once the offer is rejected, the probability that player j makes another offer in period $t + 1$ is given by p_{ij} . Thus we obtain a transition matrix $P = [p_{ij}]_{n \times n}$ indicating the probability of shifting the right to make an offer from player i to player j .

Let

$$x_{.i} = \begin{bmatrix} x_{1i} \\ x_{2i} \\ \dots \\ x_{ni} \end{bmatrix} \in \Delta^n$$

be an acceptable offer made by player i in period $t + 1$. Thus the matrix $X = [x_{.1}, \dots, x_{.n}]$ indicates all the possible offers that would be made in period $t + 1$.

Now consider player i who is making an offer in period t . Given the possible offer matrix in period $t + 1$, the expected payoff in period $t + 1$ for all the players under the transition matrix P is given by

$$E_{P_i} X = X P_i^T,$$

where P_i is the i th row of matrix P . Backward induction implies that player i 's acceptable offer in period t would be

$$x_{.i} = B_i X P_i^T, i = 1, 2, \dots, n.$$

Therefore the multilateral problem boils down to a solution to the above system of equations.

Now we introduce the notion of vec -functions for a matrix. Let $\text{vec}(X)$ be the vector obtained by listing the columns of X above one another. Then it is easily seen that

$$\text{vec}(B_i X P_i^T) = P_i \otimes B_i \text{vec}(X).$$

This leads to the following system of equations,

$$x_{.i} = P_i \otimes B_i \text{vec}(X), i = 1, 2, \dots, n.$$

The following proposition characterizes the bargaining outcome with random transition of moves.

Proposition 5 *The Markovian SPE outcome in the n -person bargaining problem with random transition P is equivalent to finding a column stochastic matrix X that satisfies the following system of equations:*

$$\begin{aligned} x_{.1} &= P_1 \otimes B_1 \text{vec}(X), \\ x_{.2} &= P_2 \otimes B_2 \text{vec}(X), \\ &\dots \\ x_{.n} &= P_n \otimes B_n \text{vec}(X). \end{aligned}$$

Denote

$$P \circledast B = \begin{bmatrix} P_1 \otimes B_1 \\ P_2 \otimes B_2 \\ \dots \\ P_n \otimes B_n \end{bmatrix}_{n^2 \times n^2}$$

as the block matrix. Then the above system of equations can be written as

$$\text{vec}(X) = [P \circledast B] \text{vec}(X).$$

Thus the problem of finding the n -person bargaining outcome with random transition is equivalent to finding a matrix $X \in \Delta^n \times \dots \times \Delta^n$ such that $\text{vec}(X) = [P \circledast B] \text{vec}(X)$.

We illustrate the above proposition with two examples.

Example 3: Two-person bargaining with random transition probabilities.

The bargaining operators are $B_1 = \begin{bmatrix} 1 & 1 - \delta_2 \\ 0 & \delta_2 \end{bmatrix}$ and $B_2 = \begin{bmatrix} \delta_1 & 0 \\ 1 - \delta_1 & 1 \end{bmatrix}$.

The transition matrix is $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$. Thus the equilibrium outcome

matrix $X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix}$ is characterized by

$$\begin{aligned} \begin{bmatrix} x_{11} \\ x_{21} \end{bmatrix} &= \begin{bmatrix} 1 & 1 - \delta_2 \\ 0 & \delta_2 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} p_{11} \\ p_{12} \end{bmatrix}, \\ \begin{bmatrix} x_{12} \\ x_{22} \end{bmatrix} &= \begin{bmatrix} \delta_1 & 0 \\ 1 - \delta_1 & 1 \end{bmatrix} \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} \begin{bmatrix} p_{21} \\ p_{22} \end{bmatrix}. \end{aligned}$$

Solving the above equations with the condition that $x_{\cdot 1}, x_{\cdot 2} \in \Delta^2$, we have a unique solution:

$$X = \begin{bmatrix} \frac{(1-\delta_2)(1-p_{22}\delta_1)}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} & \frac{\delta_1(1-\delta_2)(1-p_{22})}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} \\ \frac{\delta_2(1-\delta_1)(1-p_{11})}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} & \frac{(1-\delta_1)(1-p_{11}\delta_2)}{1-\delta_1\delta_2+p_{11}\delta_2(\delta_1-1)+p_{22}\delta_1(\delta_2-1)} \end{bmatrix}.$$

When players make alternative offers, meaning the transition is not random, and letting $p_{11} = p_{22} = 0$, the transition matrix P reduces to $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, and we obtain the classical bargaining outcome of Rubinstein:

$$X = \begin{bmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{bmatrix} = \begin{bmatrix} \frac{(1-\delta_2)}{1-\delta_1\delta_2} & \frac{\delta_1(1-\delta_2)}{1-\delta_1\delta_2} \\ \frac{\delta_2(1-\delta_1)}{1-\delta_1\delta_2} & \frac{(1-\delta_1)}{1-\delta_1\delta_2} \end{bmatrix}.$$

Example 4: Three-person bargaining problem with identical discount factor and random transition probabilities.

Here the bargaining operators are given by

$$B_1 = \begin{bmatrix} 1 & 1 - \delta & 1 - \delta \\ 0 & \delta & 0 \\ 0 & 0 & \delta \end{bmatrix}, \quad B_2 = \begin{bmatrix} \delta & 0 & 0 \\ 1 - \delta & 1 & 1 - \delta \\ 0 & 0 & \delta \end{bmatrix}, \quad B_3 = \begin{bmatrix} \delta & 0 & 0 \\ 0 & \delta & 0 \\ 1 - \delta & 1 - \delta & 1 \end{bmatrix}.$$

And

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} \\ p_{21} & p_{22} & p_{33} \\ p_{31} & p_{32} & p_{33} \end{bmatrix}.$$

The equilibrium outcome boils down to

$$x_{\cdot i} = B_i X \begin{bmatrix} p_{11} \\ p_{12} \\ p_{13} \end{bmatrix}, \quad i = 1, 2, 3.$$

We find that $X =$

$$= \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \\ x_{31} & x_{32} & x_{33} \end{bmatrix} = \begin{bmatrix} \frac{1-\delta(p_{22}+p_{33})+\delta^2(p_{22}p_{33}-p_{23}p_{32})}{D} & \frac{\delta p_{21}-\delta^2(p_{21}p_{33}-p_{31}p_{23})}{D} & \frac{\delta p_{31}-\delta^2(p_{22}p_{31}-p_{32}p_{21})}{D} \\ \frac{\delta p_{12}-\delta^2(p_{12}p_{33}-p_{13}p_{32})}{D} & \frac{1-\delta(p_{11}+p_{33})+\delta^2(p_{11}p_{33}-p_{13}p_{31})}{D} & \frac{\delta p_{32}-\delta^2(p_{11}p_{32}-p_{12}p_{31})}{D} \\ \frac{\delta p_{13}-\delta^2(p_{13}p_{22}-p_{23}p_{12})}{D} & \frac{\delta p_{23}-\delta^2(p_{11}p_{23}-p_{21}p_{13})}{D} & \frac{1-\delta(p_{11}+p_{22})+\delta^2(p_{11}p_{22}-p_{12}p_{21})}{D} \end{bmatrix}$$

where

$$D = \begin{bmatrix} 1 + \delta(1 - p_{11} - p_{22} - p_{33}) + \delta^2(1 - p_{11} - p_{22} - p_{33}) \\ + \delta^2(p_{22}p_{33} + p_{11}p_{33} + p_{11}p_{22} - p_{13}p_{31} - p_{12}p_{21} - p_{23}p_{32}) \end{bmatrix}.$$

7 Conclusion

We have shown that the Rubinstein-Stahl model of bargaining with complete information can be naturally generalized to n -person bargaining with heterogeneous discount factors. We introduce a bargaining operator for each player and a bargaining operator for the game, which is a positive column stochastic operator. We show that the efficient outcome that can be supported by a stationary Markov strategy corresponds to the unique invariant measure of the bargaining operator of the game, which is the fixed point of the bargaining operator in the simplex. A closed form of this unique SPE is provided. With this approach, we also characterize the efficient stationary outcome of multilateral bargaining with random transition matrix.

8 Appendix: The Perron-Frobenius Theorem

The following contains some results well known in matrix theory that we have used in the derivation of our results in the paper:

Let R^n be the n -dimensional Euclidean space, and R_+^n, R_{++}^n be defined as usual.

Definition 1 *i) A linear operator $A : R^n \rightarrow R^n$ is nonnegative if A leaves R_+^n invariant, i.e., $A(R_+^n) \subset R_+^n$.*

ii) The linear operator $A : R^n \rightarrow R^n$ is positive if $A(R_+^n) \subset R_{++}^n$.

iii) A nonnegative matrix $A = [a_{ij}]_{n \times n}$ is column stochastic if $\sum_i a_{ij} = 1$.

iv) Nonnegative matrix A is primitive if there is $k > 0$ such that A^k is a positive matrix.

v) A matrix A is irreducible, if it cannot be made into block upper-diagonal by simultaneous permutation of rows and columns.

The following is the well known Perron and Frobenius Theorem which is the finite dimensional version of Krein-Rutman Theorem.

Theorem 1 *Let A be a nonnegative matrix. Denote its spectrum by $\sigma(A)$. Then:*

i) The spectral radius $|\sigma(A)|$ is an eigenvalue, that is, $|\sigma(A)| \in \sigma(A)$, and is associated with a nonnegative eigenvector $x^ \in R_+^n$.*

ii) If, in addition, A is an irreducible matrix, then $|\sigma(A)| \geq |\lambda|$, for all $\lambda \in \sigma(A)$, and $|\sigma(A)|$ is a simple eigenvalue associated with a positive eigenvector $x^ \in R_{++}^n$.*

iii) If, in addition, A is a primitive matrix, then $|\sigma(A)| > |\lambda|$ for all $\lambda \in \sigma(A), \lambda \neq |\sigma(A)|$.

A particular kind of nonnegative operator is the stochastic operator. The next result is a direct application of the Perron-Frobenius Theorem.

Theorem 2 *Let A be a column stochastic operator. Then:*

- i) A leaves the simplex Δ^n invariant.*
- ii) If A is primitive, then A has a largest eigenvalue $\lambda_{\max} = 1$, and all the other eigenvalues $|\lambda| < 1$.*
- iii) When A is irreducible, it has a unique fixed point x^* in the interior of $\Delta^n = \{x \mid x_i \geq 0, \sum x_i = 1\}$.*
- iv) When A is irreducible, $x^k = A^k x$ converges to the unique fixed point x^* in Δ , for any given initial $x \in \Delta^n$.*

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